

# A New Method to Real-Normalize Measured Complex Modes

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A time domain subspace iteration technique is presented to compute a set of normal modes from the measured complex modes. By using the proposed method, a large number of physical coordinates are reduced to a smaller number of modal or principal coordinates. Subspace free decay time responses are computed using properly scaled complex modal vectors. Companion matrix for the general case of nonproportional damping is then derived in the selected vector subspace. Subspace normal modes are obtained through eigenvalue solution of the  $[M_N]^{-1}[K_N]$  matrix and transformed back to the physical coordinates to get a set of normal modes. A numerical example is presented to demonstrate the outlined theory.

## INTRODUCTION

Experimental modal analyses are carried out to extract a set of modal parameters from the measured time or frequency domain data of the structure under test. These identified eigenvectors are in general damped complex modes due to several possible reasons<sup>[1,2]</sup>:

1. The damping is nonproportional, i.e.,  $[C]$  matrix is not proportional to  $[K]$  and  $[M]$  matrices.
2. Measurement errors due to mass loading effects, noise, nonlinearities, etc.
3. Digital signal processing errors due to finite frequency resolution, leakages, high modal density, and frequency response functions estimation procedure ( $H_1, H_2, H_v$ ).
4. Modal parameter estimation errors due to invalid estimation of number of degrees of freedom.

Identified complex modes can be used directly in the applications of modal modeling, structural dynamic modification and sensitivity analysis<sup>[3,4]</sup>, or validation and optimization of an analytical model<sup>[5,6]</sup>. On the other hand, real normal modes are sometimes more desired in the similar applications due to the facts that (i) normal modes are numerically easier to handle than complex modes and (ii) analysts usually compute normal modes rather than complex modes in the finite element analysis due to the lack of information of physical damping matrix. If normal modes are desired from a set of identified complex modes, a real-normalization procedure is needed to be implemented. Several methods have been proposed<sup>[7-13]</sup> in the past to derive normal modes from identified complex modes. In this paper, a time domain subspace iteration technique (an improved approach from the method described in Reference 12) developed by the Structural Dynamics Research Laboratory (SDRL) is proposed to real-normalize identified complex modes originated from systems with nonproportional damping.

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## THEORY AND FORMULATION: in the physical coordinates

Assume there exists a set of measured modal parameters from a modal test. These modal parameters consists of  $N$  complex modes (and their complex conjugates),  $\lambda_r$ ,  $r=1, \dots, N$  and  $\{\psi\}_r$ ,  $r=1, \dots, N$ . Each modal vector has dimensions  $N_m$  and in general  $N_m \gg N$ . To compute a set of normal modes from the given complex modes, a time-domain approach developed in the physical coordinates is formulated.

### Formulation of Free Decay Time Responses

For the given modal parameters, displacement, velocity and acceleration responses can be expressed as

$$\{x(t)\} = \sum_{r=1}^{2N} \{\psi\}_r e^{\lambda_r t} \quad (1)$$

$$\{\dot{x}(t)\} = \sum_{r=1}^{2N} \lambda_r \{\psi\}_r e^{\lambda_r t} \quad (2)$$

$$\{\ddot{x}(t)\} = \sum_{r=1}^{2N} \lambda_r^2 \{\psi\}_r e^{\lambda_r t} \quad (3)$$

### Computation of Companion Matrix [E]

The equations of motion for the general case of nonproportional damping is

$$\begin{bmatrix} [C] & [M] \\ [M] & [0] \end{bmatrix} \begin{Bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{Bmatrix} + \begin{bmatrix} [K] & [0] \\ [0] & -[M] \end{bmatrix} \begin{Bmatrix} x(t) \\ \dot{x}(t) \end{Bmatrix} = \begin{Bmatrix} [f] \\ [0] \end{Bmatrix} \quad (4)$$

Another form of the homogeneous Equation 4 can be written as

$$\begin{Bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{Bmatrix} = \begin{bmatrix} [0] & [I] \\ -[M]^{-1}[K] & -[M]^{-1}[C] \end{bmatrix} \begin{Bmatrix} x(t) \\ \dot{x}(t) \end{Bmatrix} \quad (5)$$

Let

$$[E] = \begin{bmatrix} [0] & [I] \\ -[M]^{-1}[K] & -[M]^{-1}[C] \end{bmatrix} \quad (6)$$

Equation 5 can then be written as

$$\{\dot{y}(t)\} = [E] \{y(t)\} \quad (7)$$

where  $\{y(t)\}$  is now the system's state vector containing the displacement response  $\{x(t)\}$  and velocity responses  $\{\dot{x}(t)\}$

and  $[E]$  is called the companion matrix. In the physical coordinates, by repeating Equation (7)  $2N_m$  time instants and dropping vector notation, the following equation is satisfied

$$[\dot{y}(t_1), \dots, \dot{y}(t_{2N_m})] = [E][y(t_1), \dots, y(t_{2N_m})] \quad (8)$$

where  $t_1, t_2, \dots, t_{2N_m}$  represent  $2N_m$  time instants. From Equation 8, the companion matrix  $[E]$  can be computed

$$[E] = [\dot{Y}][Y]^{-1} \quad (9)$$

where  $[E]$ ,  $[Y]$  and  $[\dot{Y}]$  are all  $2N_m \times 2N_m$  matrices.

### Computation of Normal Modes

By computing the companion matrix  $[E]$ ,  $[M]^{-1}[K]$  matrix gives the normal modes according to the eigenvalue solution

$$[[M]^{-1}[K]] [\Phi] = \omega_r^2 [\Phi], \quad r = 1, \dots, N \quad (10)$$

In the physical coordinate,  $[Y]$  is always singular due to the number of measurement degrees of freedom  $N_m$  is much larger than the number of measured modes  $N$ . In other words, it is numerically difficult and unstable to solve  $[E]$  in the physical coordinates<sup>[7,11]</sup>. Therefore, it is proposed to compute the companion matrix  $[E]$  in the principal or modal coordinates in the following way.

### NORMAL MODES SOLUTION: subspace iteration technique

Based on a set of identified complex modes, a time domain subspace iteration technique is developed to obtain a set of normal modes in the following way:

#### Hypothesis

Define  $[\hat{\Phi}]$  = transformation matrix of the identified complex modal matrix  $[\Psi]$ , it is assumed that  $[\Psi]$  can be expressed as

$$[\Psi] = [\hat{\Phi}][W] \quad (11)$$

where  $[W]$  is a  $N \times N$  complex matrix and can be obtained through pseudo inverse technique

$$[W] = [\hat{\Phi}]^+ [\Psi] \quad (12)$$

For the proposed method, the selection of the transformation matrix  $[\hat{\Phi}]$  can not be arbitrary since it represents a N-dimensional subspace of the physical  $N_m$ -dimensional vector space from which the companion matrix  $[E_N]$  and the normal modes are derived. Therefore, the following method to determine the transformation matrix  $[\hat{\Phi}]$  is proposed and justified.

### Determination of the Transformation Matrix $[\hat{\Phi}]$

Using the proposed method, it can be shown that choice of the  $[\hat{\Phi}]$  has significant effects on the calculated normal modes. Since the transformation matrix  $[\hat{\Phi}]$  can be considered as a N-dimensional vector subspace (column space of  $[\hat{\Phi}]$ ), into which the physical system matrices and the identified complex modes are transformed, therefore, it must be so chosen such that the following conditions are satisfied:

1.  $[\hat{\Phi}]$  must not be orthogonal to the normal modal matrix  $[\Phi]$  of the undamped system, and
2.  $[\hat{\Phi}]$  should be selected that it is as close to  $[\Phi]$  as possible.
3.  $[\hat{\Phi}]$  must has rank N.

A logical way to select  $[\hat{\Phi}]$  is based on the identified complex modal matrix  $[\Psi]$ .

1. Fillod<sup>[14]</sup> proved that if the complex modal vector is normalized according to the following equation

$$2\lambda_r\{\psi\}_r^T[M]\{\psi\}_r + \{\psi\}_r^T[C]\{\psi\}_r = 2j\omega_r \quad (13)$$

,the imaginary part of  $\{\psi\}_r$  is minimized. This indicates that the real part of  $\{\psi\}_r$  is maximized and contains maximum useful information of the identified complex modes.

2. It can be proved (see Appendix) that if a undamped system is perturbed to the first order by a nonproportional damping matrix, and the calculated complex modes are normalized according to Equation 13, then the real parts of the normalized complex modes are very close to the normal modes of the undamped system.
3. The transformation matrix obtained by taking the real part of a set of complex modes normalized according to Equation 13, has rank N due to the fact that (i) the identified complex modes are independent of each other, and (ii) norms of the columns of  $[\hat{\Phi}]$  have the same order of magnitude, i.e, eigenvalues found in Equation 19 using singular value decomposition technique also have the same order of magnitude.

From the above observations, it can be concluded that real part of the complex modes normalized to  $2j\omega$ , according to Equation 13 is the best choice of the transformation matrix  $[\hat{\Phi}]$ .

### Computation of Free Decay Time Responses in Subspace

In order to derive the companion matrix  $[E_N]$  in the selected subspace  $[\hat{\Phi}]$ , free decay time responses in the physical coordinates needs to be transformed into the selected vector subspace using one of the following two approaches:

#### Modal Space Approach

The displacement vectors in the physical and modal coordinate systems are related by the following transformation:

$$\{x(t)\} = [\hat{\Phi}] \{p(t)\} \quad (14)$$

or

$$\{p(t)\} = [\hat{\Phi}]^+ \{x(t)\} \quad (15)$$

where  $[\hat{\Phi}]^+$  is the pseudo inverse of  $[\hat{\Phi}]$ .

Substitute Equation 1 and 11 into 15, for the given modal parameters, free decay displacement in the modal coordinate can be written as

$$\begin{aligned} \{p(t)\} &= [\hat{\Phi}]^+ 2 \operatorname{Re}\{[\Psi]\{e^{\lambda t}\}\} \\ &= 2 [\hat{\Phi}]^+ [\Phi] \operatorname{Re}\{[W]\{e^{\lambda t}\}\} \\ &= 2 \operatorname{Re}\{[W]\{e^{\lambda t}\}\} \end{aligned} \quad (16)$$

where "Re" represents the real part of a matrix.

Similarly, velocity and acceleration responses in the modal coordinates can be expressed as

$$\{\dot{p}(t)\} = 2 \operatorname{Re}\{[W]\{\lambda e^{\lambda t}\}\} \quad (17)$$

$$\{\ddot{p}(t)\} = 2 \operatorname{Re}\{[W]\{\lambda^2 e^{\lambda t}\}\} \quad (18)$$

### Principal Response Analysis (PRA) Approach

Using singular value decomposition technique and assuming  $[\hat{\Phi}]$  has full rank  $N$ ,  $[\hat{\Phi}]$  can be decomposed as

$$[\hat{\Phi}] = [P] {}^T \Sigma [S]^H \quad (19)$$

where

- $[P]$  = orthonormal matrix ( $N_m \times N$ )
- ${}^T \Sigma$  = diagonal matrix consists of  
eigenvalues of  $[\hat{\Phi}]^H [\hat{\Phi}]$  ( $N \times N$ )
- $[S]$  = unitary matrix consists of  
eigenvectors of  $[\hat{\Phi}]^H [\hat{\Phi}]$  ( $N \times N$ )

Using matrix  $[P]$  as the transformation matrix, displacement in the physical coordinates can be transformed into the principal coordinates according to

$$\{x(t)\} = [P] \{p(t)\} \quad (20)$$

or

$$\{p(t)\} = [P]^H \{x(t)\} \quad (21)$$

Substitute Equation 1 and 19 into Equation 21,  $\{p(t)\}$  can be expressed as

$$\begin{aligned} \{p(t)\} &= [P]^H 2 \operatorname{Re}\{\Psi\} \{e^{\lambda t}\} \\ &= 2 [P]^H [\hat{\Phi}] \operatorname{Re}\{[W]\} \{e^{\lambda t}\} \\ &= 2 [P]^H [P]^T \Sigma_j [S]^H \operatorname{Re}\{[W]\} \{e^{\lambda t}\} \\ &= 2^T \Sigma_j [S]^H \operatorname{Re}\{[W]\} \{e^{\lambda t}\} \end{aligned} \quad (22)$$

Similarly, velocity and acceleration responses can be written as

$$\{\dot{p}(t)\} = 2^T \Sigma_j [S]^H \operatorname{Re}\{[W]\} \{\lambda e^{\lambda t}\} \quad (23)$$

$$\{\ddot{p}(t)\} = 2^T \Sigma_j [S]^H \operatorname{Re}\{[W]\} \{\lambda^2 e^{\lambda t}\} \quad (24)$$

### Computation of companion matrix $[E_N]$

Similar to the derivations (Equation 4 to Equation 6) in the physical coordinates, the companion matrix  $E_N$  can be computed in the selected vector subspace using the time responses in equations 16-18 (modal space approach) or 22-24 (PRA approach)

$$\{q(t)\} = [E_N] \{\dot{q}\} \quad (25)$$

where  $\{q(t)\}$  is now the system's state vector in the selected subspace containing the displacement response  $\{p(t)\}$  and velocity responses  $\{\dot{p}(t)\}$ , and

$$[E_N] = \begin{bmatrix} [0] & [I] \\ -[M_N]^{-1}[K_N] & -[M_N]^{-1}[C_N] \end{bmatrix} \quad (26)$$

where  $[M_N]$ ,  $[K_N]$  and  $[C_N]$  are the reduced system mass, stiffness and damping matrix respectively and can be written as

$$[M_N]_{N \times N} = [\hat{\Phi}]^T [M] [\hat{\Phi}] \quad (\text{modal space}) \quad (27a)$$

$$[M_N]_{N \times N} = [P]^H [M] [P] \quad (\text{PRA}) \quad (27b)$$

$$[K_N]_{N \times N} = [\hat{\Phi}]^T [K] [\hat{\Phi}] \quad (\text{modal space}) \quad (28a)$$

$$[K_N]_{N \times N} = [P]^H [K] [P] \quad (\text{PRA}) \quad (28b)$$

$$[C_N]_{N \times N} = [\hat{\Phi}]^T [C] [\hat{\Phi}] \quad (\text{modal space}) \quad (29a)$$

$$[C_N]_{N \times N} = [P]^H [C] [P] \quad (\text{PRA}) \quad (29b)$$

Once  $\{q(t)\}$  and  $\{\dot{q}(t)\}$  are computed at  $2N$  time instants, companion matrix  $[E_N]_{2N \times 2N}$  can then be computed in the modal or principal coordinates accordingly

$$[\dot{q}(t_1), \dots, \dot{q}(t_{2N})] = [E_N][q(t_1), \dots, q(t_{2N})] \quad (30)$$

and

$$[E_N] = [\dot{Q}][Q]^{-1} \quad (31)$$

Since  $[Q]$  is a matrix with full rank  $2N$ , it is always invertible.

### Computation of Normal modes: first iteration

From the companion  $[E_N]$ , the  $N$  by  $N$   $[M]^{-1}[K]$  matrix gives the normal mode solutions  $[\Phi]$  in the modal or principal coordinates according to the eigenvalue solution

$$[[M_N]^{-1}[K_N]] [\Phi] = \Omega_r^2 [\Phi], \quad r = 1, \dots, N \quad (32)$$

where  $\Omega_r$  is the  $r$ -th undamped natural frequency.

From the modal space method, it is noticed that (i) if  $[\hat{\Phi}]$  is selected as the normal modal matrix  $[\Phi]$  of the undamped system, then  $[M_N]$ ,  $[K_N]$  and consequently,  $[M_N]^{-1}[K_N]$  are all diagonal matrices by the orthogonality conditions, and (ii) Generalized damping matrix  $[C_{gen}]$  can also be obtained from the companion matrix  $[E_N]$  as  $[M_N]^{-1}[K_N]$ . Generalized damping matrix is defined as

$$[C_{gen}] = [\Phi]^T [C] [\Phi] \quad (33a)$$

and  $[\Phi]$  must also satisfies the following relationship

$$[\Phi]^T [M] [\Phi] = [I] \quad (33b)$$

First estimation of the normal modes  $\{\psi\}_r, r=1, \dots, N$  of the undamped system can then be obtained through the coordinate transformation

$$[\Phi_1] = [\hat{\Phi}][\Phi] \quad (\text{modal space}) \quad (34a)$$

or

$$[\Phi_1] = [P][\Phi] \quad (PRA) \quad (34b)$$

where  $[\Phi_1]$  is the first estimation of the normal modal matrix.

## Subspace Iteration Technique

In order to improve (if possible) and check the accuracy of  $[\Phi_1]$ ,  $[\Phi_1]$  is compared with  $[\hat{\Phi}]$  column by column to check the convergence. If not convergent, then  $[\Phi_1]$  will substitute  $[\Phi]$  in Equation 11, 14 and 19 and a new iteration will start to find a  $[\Phi_2]$ . This process continues until  $[\Phi_m]$  converges to  $[\Phi_{m-1}]$ , where  $m$  is the number of iterations. In general,  $m$  is usually 2 or 3, i.e., the transformation matrix  $[\hat{\Phi}]$  converges very fast because all iterations are done in the same vector subspace which is defined by the initially estimated transformation matrix  $[\hat{\Phi}]$ .

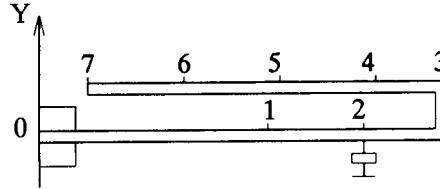


Figure 1. Folded Beam Used as Test Structure

## CASE STUDY

The proposed method is applied to the identified complex modes of a simple structure as shown in Figure 1 (For comparisons, this example is the same as the one presented in Reference 11). This structure is a folded beam excited on its bending modes along the  $y$  axis. Its dynamic characteristics shows the existence of pairs of eigenvectors at very closely spaced eigenvalues (quasi multiplicity of order 2). The eigenvectors are described by seven measurement degrees of freedom as indicated in Figure 1. The first two complex modes are identified from a set of frequency response functions measured using slow sine sweep excitation technique. The identified complex modes scaled according to Equation 13 are listed in Table 1. The companion matrices  $[E_N]$  calculated in the selected subspace using both the modal space and the PRA approaches are listed in Table 2. Two undamped natural frequencies and their corresponding normal modes based on the proposed method and the three methods proposed in Reference 11 are listed in Table 3. Modal Assurance Criteria (MAC)<sup>[15]</sup> are computed for these two normal modes between those obtained from the proposed method and those obtained from the methods described in Reference 11.

From Table 3, both the modal space and the PRA approaches calculate the same undamped natural frequencies and normal modes of the associated undamped system.

From Table 2, companion matrix  $[E_N]$  obtained from the second iteration of the modal space approach shows (i) the  $[M_N]^{-1}[K_N]$  matrix is fairly diagonalized and (ii) the  $[M_N]^{-1}[C_N]$  matrix is fairly symmetrical. These indicate that the calculated normal modes are very close to the true real modes of the associated undamped system. Therefore, the calculated  $[M_N]^{-1}[C_N]$  matrix listed in Table 5 is also very close to the generalized damping matrix derived in Reference 11.

From Table 3 and 4, two normal modes derived from the proposed method stay very close to those derived in Reference 11.



**TABLE 1. Measured Complex modes**

Measured Complex modes				
Point No.	First Mode		Second Mode	
	$\lambda_1 = -1.992 + j\ 209.55$		$\lambda_2 = -2.547 + j\ 213.40$	
	real	imaginary	real	imaginary
1	-0.144	0.118	0.266	.0683
2	-0.303	0.203	0.457	0.138
3	-0.498	0.267	0.598	0.228
4	-0.179	0.231	0.519	0.0847
5	0.210	0.207	0.464	-0.0942
6	0.618	0.192	0.433	-0.280
7	1.060	0.192	0.416	-0.483

**TABLE 2. Calculated Companion Matrix  $[E_N]$**

Companion matrix $[E_N]$								
Iteration No.	Modal Space Approach				PRA Approach			
1	0.0000	0.0000	1.0000	0.0000	0.0000	-.0001	1.0000	0.0000
	0.0000	0.0001	0.0000	1.0000	0.0000	0.0000	0.0000	1.0000
	-43507.2	-120.312	-3.7156	4.3521	-43970.09	-1182.15	-0.8233	2.9472
	-117.281	-45973.5	4.3826	-5.3629	-796.414	-45510.59	2.0166	-8.2553
2	0.0000	-.0001	1.0000	-.0000	-.0000	0.0001	1.0000	0.0000
	0.0000	0.0000	0.0000	1.0000	0.0000	-.0000	0.0000	1.0000
	-43501.49	0.0000	-4.1383	4.3492	-43921.51	-1142.37	-0.9521	3.1301
	-0.0156	-45979.18	4.5045	-4.9402	-756.586	-45559.18	2.1995	-8.1294

**TABLE 3. Calculated Normal Modes**

Calculated Normal Modes										
	First Mode					Second Mode				
	Method 1	Method 2	Method 3	Modal Spa.	PRA	Method 1	Method 2	Method 3	Modal Spa.	PRA
	natural frequency (rad/sec)					natural frequency (rad/sec)				
	Point No.	208.38	208.59	208.68	208.58	214.22	214.45	214.32	214.43	214.43
1		.161	.153	.153	.151	.151	.458	.454	.454	.451
2		.333	.312	.318	.312	.312	.774	.769	.771	.771
3		.535	.506	.515	.506	.506	1.0	1.0	1.0	1.0
4		.215	.198	.200	.196	.196	.905	.890	.892	.889
5		.169	.179	.178	.181	.181	.867	.830	.836	.827
6		.571	.572	.577	.574	.574	.873	.810	.826	.815
7		1.0	1.0	1.0	1.0	1.0	.901	.82	.832	.815

TABLE 4. MAC for the Calculated Normal Modes

Modal Assurance Criteria						
	First Mode			Second Mode		
	Method 1	Method 2	Method 3	Method 1	Method 2	Method 3
PRA	.999122	.9999906	.9999941	.998605	.999992	.999913

TABLE 5. Generalized Damping Matrix [ $C_{gen}$ ]

Method 2		Method 3		Proposed Method	
4.86	-4.72	4.14	-4.27	4.14	-4.35
-4.72	5.50	-4.27	4.94	-4.50	4.94

## CONCLUSION

The proposed method described in this paper has the following advantages:

1. This method is numerically very efficient and stable because all computations are performed in the selected vector subspace and it requires very few iterations.
2. Since this method uses a time domain approach, it can be proved that the calculated normal modes are not sensitive to the norm or scaling errors existing in the identified complex modes.

The drawbacks of this method are:

1. It can not improve the norm or scaling errors existing in the identified complex modes.
2. The calculated normal modes are subject to modal truncation errors since an incomplete set of modes are identified.

It can be concluded that the success of the proposed method is dependent on the selection of the transformation matrix  $[\hat{\Phi}]$  and the quality of the measured complex modes. It seems to be a reliable method that can be used as a post-processing procedure of the identified complex modes for further applications.

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## APPENDIX

The characteristics equations of a undamped and a damped system can be written as

$$[K][\Phi] - [M][\Phi] \Omega^2 = 0. \quad (35)$$

$$[M][\Psi] \Lambda^2 + [C][\Psi] \Lambda + [K][\Psi] = 0 \quad (36)$$

It can be proved that in the case of a lightly damped system, i.e,  $[C] = \varepsilon [\underline{C}]$ , where  $\varepsilon$  is a small perturbation parameter, the natural frequencies and the normal modes of the associated undamped system can be approximated by taking the damped natural frequencies and the real part of the complex modal vectors respectively.

Assuming:

$$[\Phi]^T [\underline{C}] [\Phi] = [\Gamma] \quad (37)$$

$$\Lambda_j = j \Omega_j + \varepsilon U_j \quad (38)$$

$$[\Psi] = [\Phi] \left[ [I] + \varepsilon j [V] \right] \quad (39)$$

and recalling that complex modal vectors satisfy the following orthogonality conditions: (Note that both the Equation 13 and 40 are scaled to  $2j\omega$ )

$$[\Psi]^T[M][\Psi]\omega_j + \omega_j[\Psi]^T[M][\Psi] + \varepsilon[\Psi]^T[C][\Psi] = 2j\omega_j \quad (40)$$

$$[\Psi^*]^T[M][\Psi]\omega_j + \omega_j^*[\Psi^*]^T[M][\Psi] + \varepsilon[\Psi^*]^T[C][\Psi] = 0 \quad (41)$$

where "\*" represents the complex conjugate of a matrix.

Substitute Equation 35,36 and 37 into Equation 40 and 41, equating the first order terms of  $\varepsilon$ :

$$[\Gamma] = -2[U_j + \Omega_j[V + V^T] + [V + V^T]\Omega_j] \quad (42)$$

$$[\Gamma] = -2[U_j + \Omega_j[V - V^T] + [V - V^T]\Omega_j] \quad (43)$$

and then

$$[\Gamma] = -2[U_j + [V]\Omega_j + \Omega_j[V]^T] \quad (44)$$

$$0 = \Omega_j[V] + [V]^T\Omega_j \quad (45)$$

From Equation 45, the matrix  $\Omega_j[V]$  is anti-symmetrical, therefore the diagonal elements of this matrix are all zero.

Since  $[\Gamma]$  is a real matrix. From Equation 44, it can be concluded that:

1.  $[U_j]$  is a real diagonal matrix. This indicates that the effect of the first order perturbation damping matrix on the eigenvalues is purely real.
2.  $[V]$  is a real matrix. This indicates that the effect of the first order perturbation damping matrix on the eigenvectors is purely imaginary.